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List of exercises I

101. **(warm up)** Throw a 6-sided die, write down the σ-algebra generated by 1 and 4. (Interpretation: if you can answer 'did I get a 1' and 'did I get a 4', you can answer all the yes/no questions in this σ-algebra)

102. Let τ_1, τ_2 be $\mathcal{F}_t$-stopping times. Argue that

    (a) $\tau_1 \wedge \tau_2$ is an stopping time

    (b) $\tau_1 \vee \tau_2$ is an stopping time

    (c) $\tau_1 + \tau_2$ is an stopping time

    (d) $\tau_1 - \tau_2$ is not necessarily an stopping time (give a counter example)

103. Show that $Cov(B_s, B_t) = s \wedge t$ where B is a standard Brownian motion.

104. **(Brownian scaling)** Let B_t be a 1-d standard Brownian motion and let $c > 0$ be a constant. Show that

$$\tilde{B}_t := \frac{1}{c} B_{c^2 t}$$

is also a standard Brownian motion. Give your intuition of this property.

105. **(Moments of BM)** We know that $B_t \sim \mathrm{N}(0, t)$ This implies that (by the characteristic function of a Gaussian)

$$\mathsf{E}[e^{iuB_t}] = e^{-\frac{1}{2}u^2 t}.$$

Apply the power series expansion of the exponential function on both sides, compare the terms with the same power of u and show that

$$\mathsf{E}[B_t^4] = 3t^2.$$

More generally, with tiny bit more effort, one can show that for $k \in \mathbb{N}$,

$$\mathsf{E}[B_t^{2k}] = \frac{(2k)!}{2^k k!} t^k.$$

(We will solve this problem later in this module in a different way.)

106. **(Moments of BM again)** Let $W_t \in \mathbb{R}$, $W_0 = 0$ be a Brownian motion. Define

$$\beta_k(t) = \mathbb{E}[W_t^k], \quad k = 0, 1, \ldots, t \geq 0.$$

    i) Use Ito's formula to show that

$$\beta_k(t) = \frac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s) ds, \quad k \geq 2.$$

ii) Show that

$$\mathbb{E}[W_t^{2k+1}] = 0$$

and

$$\mathbb{E}[W_t^{2k}] = \frac{(2k)!t^k}{2^k k!}$$

for $k = 1, 2, \ldots$.

201. **(Martingales)** Show that the following stochastic processes are martingales:
   (a) $M_t = B_t^2 - t$, (Hint: Write $B_t = B_t - B_s + B_s$, for $s < t$)
   (b) $M_t = \mathsf{E}[X|\mathcal{F}_t]$, ("Doob's m.g.". It can be thought of as the evolving sequence of best approximations to the random variable X based on accumulated information.)
   (c) $M_t = t^2 B_t - 2\int_0^t sB_s ds$,
   (d) $M_t = B_1(t)B_2(t)$, where (B_1, B_2) is a 2-dim standard BM.

202. ***(Construction of the Ito integral)** Let B be a standard Brownian motion. Find a sequence of piecewise constant process $\phi_n(t, \omega)$ such that

$$\mathbb{E}[\int_0^t (\phi_n(s) - B(s))^2 ds] \to 0.$$

Compute $\int_0^t \phi_n(s)dB_s$ and show that it converges (in what sense?) to $\frac{1}{2}(B_t^2 - t)$ if we consider finer and finer partitions. Deduce that

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t).$$

203. **(Continuation of 202)** What does the result of 202 tell you? How does it relate to problem 201.a?

204. **(Stratonovich integral)** When using $B_{t_{j^*}}$ to approximate B_t on an interval $[t_j,\ t_{j+1}]$, if $t_{j^*} = t_j$ it yields an Ito integral. Now take $t_{j^*} = \frac{1}{2}(t_j + t_{j+1})$, define the Stratonovich integral by

$$\int_0^T X(t, \omega) \circ dB_t(\omega) = \lim_{\Delta t_j \to 0} X(t_{j^*}, \omega)\Delta B_j$$

Use the example $\int_0^T B_t \circ dB_t$ to see that the Stratonovich integral follows the usual chain rule. Compare to exercise 202, does Ito follow the normal chain rule?

205. Argue that the Ito integral $\int_0^t s dB_s$ is a normal random variable. Show that it follows the distribution $N(0, \frac{1}{3}t^2)$.

   Remark. In fact, integrate any deterministic function $f(t)$ with respect to the Brownian motion yields a Gaussian process

$$I_t(f) = \int_0^t f(s)dB_s \sim N(0, \int_0^t f^2(s)ds).$$

The exercises in section 3 contains quite some hands-on training on using Ito's formula (301-305). You can skip them if you are already familiar with Ito calculus.

In the following exercises, When we say a geometric brownian motion (GBM) we refer to the process

$$dX_t = \mu X_t dt + \sigma X_t dB_t, X_0 = x.$$

301. Use Ito's formula to show that the expectation of a GBM satisfies

$$\mathbb{E}[X_t] = xe^{\mu t}.$$

302. Use Ito's formula to calculate $\int_0^t B_s^2 dB_s$.

303. Let α_t be some L^2 stochastic process, and define the 1d stochastic process

$$Z_t = \exp(\int_0^t \alpha_s dB_s - \frac{1}{2}\int_0^t \alpha_s^2 ds).$$

Use Ito's formula to show that $dZ_t = \alpha_t Z_t dB_t$, and conclude that it is a martingale (given that $\alpha_t Z_t \in L^2$). (**Remark.** This exercise provides a way to construct martingales.)

304. Solve the 1d SDE $dX_t = X_t dt + dB_t$ with $X_0 = x$ (Hint: multiply both sides with an 'integrating factor' e^{-t}).

305. Solve the 1d SDE $dY_t = rdt + \alpha Y dB_t$, where $Y_0 = 0, r, \alpha \in \mathbb{R}$. (Hint: multiply both sides with an 'integrating factor' $F_t = \exp(-\alpha B_t + \frac{1}{2}\alpha^2 t)$).

306. **(Brownian bridge)** Let X_t be the solution to the following SDE with $X_0 = 0$:

$$dX_t = -\frac{1}{1-t} X_t dt + dW_t, \quad 0 \le t < 1.$$

   i) Use Ito's lemma to show that $X_t = (1-t)\int_0^t \frac{dW_s}{1-s}$ solves the above SDE. *(Hint: consider the process $Y_t = \int_0^t \frac{dW_s}{1-s}$.)*

   ii) Argue that for $t \in [0, 1)$, X_t is a Gaussian random variable with mean 0 and variance $t(1-t)$. Show further that X_t converges to 0 in L^2 as $t \to 1$:

$$\lim_{t \uparrow 1} \mathbb{E}[X_t^2] = 0.$$

   (Remark: the process X_t is pinned on both ends, hence a "bridge")

307. **(SDE on a circle)** Consider the following equation

$$dX_t = -\frac{1}{2} X_t dt - Y_t dW_t,$$
$$dY_t = -\frac{1}{2} Y_t dt + X_t dW_t.$$

Let $(X_0, Y_0) = (x, y)$ such that $x^2 + y^2 = 1$. Show that $X_t^2 + Y_t^2 = 1$ for all t and thus this SDE lives on the unit circle.

308. **(Hyperbolic SDE)** Similarly, consider the following equation

$$dX_t = \frac{1}{2}X_t dt + Y_t dW_t,$$
$$dY_t = \frac{1}{2}Y_t dt + X_t dW_t.$$

Show that $X_t^2 - Y_t^2$ is constant for all t.

309. Let X_t, Y_t be two one-dimensional Ito processes, prove the following Ito product rule:

$$dX_tY_t = X_t dY_t + Y_t dX_t + dX_t dY_t.$$

310. ***(Complex BM)** Given a two-dimensional Brownian motion $(B^{(1)}, B^{(2)})$, define the complex Brownian motion

$$B_t^c := B_t^{(1)} + iB_t^{(2)}.$$

Let $f : \mathbb{C} \to \mathbb{C}$ be a function of the form $f(z) = f_{\mathbb{R}}(z) + if_{\mathbb{C}}(z)$, for any $z \in \mathbb{C}$, and $f_{\mathbb{R}}, f_{\mathbb{C}} : \mathbb{C} \to \mathbb{R}$. If f is is analytic, i.e. satisfies the Cauchy-Riemann equations:

$$\frac{\partial f_{\mathbb{R}}}{\partial x} = \frac{\partial f_{\mathbb{C}}}{\partial y}, \quad \frac{\partial f_{\mathbb{R}}}{\partial y} = -\frac{\partial f_{\mathbb{C}}}{\partial x},$$

where $z = x + iy$. Show that the identity $df(B_t^c) = f'(B_t^c)dB_t^c$ holds, where f' denotes the complex derivative of f.

Module 2

401. Find the infinitesimal generator for the following Ito processes:

   (a) $dX_t = \beta X_t dt + \sigma dB_t$,

   (b) $dX_t = \mu X_t dt + \sigma X_t dB_t$,

   (c) the n-dimensional Brownian motion,

   (d) the (n+1)-dimensional stochastic process (t, B_t), where $B_t \in \mathbb{R}^n$.

402. Find an Ito process whose generator is the following:

   (a) $\mathcal{L}f(x) = f'(x) + f''(x)$,

   (b) $\mathcal{L}f(x) = rf'(x) + \frac{1}{2}\alpha x^2 f''(x)$.

403. Prove the 1d general Feynman-Kac theorem when $D = \mathbb{R}$, and $r > 0$ being a constant. (Hint: apply Ito's formula on $Y_s = e^{-r(s-t)}u(s, X_s)$).

404. Use Feynman-Kac to solve the PDE with terminal condition:

$$\begin{cases} u_t + \frac{1}{2}u_{xx} = 0, \\ u(x, T) = x^4. \end{cases}$$

405. Use Feynman-Kac to solve the PDE with initial condition:

$$\begin{cases} u_t - \mu x u_x - \frac{1}{2}\sigma^2 u_{xx} = 0, \\ u(0,x) = \Phi(x). \end{cases}$$

(Hint: take $v(t,x) = u(T-t,x)$)

406. Let $\tau = \inf\{t > 0 : B_t \notin (-a,b)\}$, where $a, b > 0$. Use Dynkin's formula to determine $\mathbb{E}[\tau]$.

407. Let $J = \int_0^1 t dB_t$ Use Dynkin's formula to find the moment generating function $m(u) = \mathbb{E}[e^{uJ}]$, and show that $J \sim N(0, \frac{1}{3})$. (Hint: let $f(x) = e^{ux}$)

408. **(The Ornstein-Uhlenbeck process)** For a given standard Brownian motion W on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the Ornstein-Uhlenbeck process which solves the following SDE:

$$dX_t = \mu X_t dt + \sigma dW_t, \quad X_0 = x, \tag{1}$$

where $\mu, \sigma \in \mathbb{R}$.

i) Show that equation (1) admits a unique strong solution.

ii) Use Ito's formula to find the solution to the above equation *(Hint: consider the process $e^{-\mu t}X_t$)*.

iii) Fix $T \geq 0$, find $\mathbb{E}[X_T]$ and $Var[X_T]$.

iv) We wish now to compute the characteristic function ϕ of X_T:

$$\phi_{X_T}(\xi) := \mathbb{E}[e^{i\xi X_T} | X_0 = x], \text{ for } \xi \in \mathbb{R}.$$

Fix $\xi \in \mathbb{R}$, use the Feynman-Kac theorem to show that the function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ defined by $u(t,x) := \mathbb{E}[e^{i\xi X_T} | X_t = x]$ satisfy the following PDE:

$$u_t + \mu x u_x + \frac{1}{2}\sigma^2 u_{xx} = 0, \text{ for all } (t,x) \in [0,T) \times \mathbb{R}.$$

Determine the terminal condition $u(T,x)$.